Positive Realizability on Horn Filters

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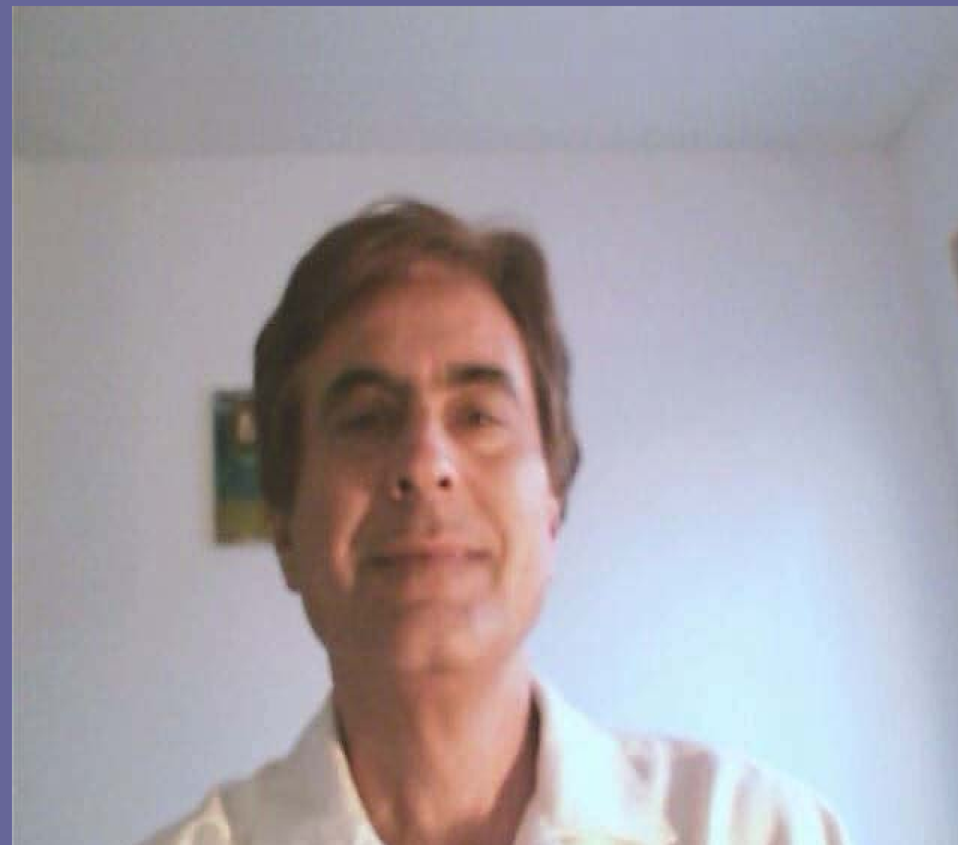

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Positive Realizibility on Horn Filters

During 2005-2007 the author explored positive and Horn fragment categories based on his 1981 on positive forcing on Kiesler fragments. Positive forcing had defined T* on a theory T to be T augmented with induction schemas on the generic diagram functions. for a model M for T. Let P be a poset, F a family of sets, G subset of P.The author defined positive local realizability on 2005-2007 publications (ASL).

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The Keisler Fragment

Start with a well-behaved countable fragment of an infinitary language $L\omega 1,\omega$ defined by H.J. Keisler. The term fragment is not inconsistent with the terminology applied in categorical logic. A subclass F of class of all formulas of an Infinitary language is called a fragment, if (a) for each formula φ in F all the subformulas of φ also belong to F; and (b) F is closed under substitution: if φ is in F, t is a term of L, x is a free variable in φ, the $\varphi(x/t)$ is in F.

Definition By a fragment of $L\omega 1,\omega$ we mean a set $L\langle A\rangle$ of formulas such that
(1) Every formula of L belongs to $L\langle A\rangle$
(2) $L\langle A\rangle$ is closed under $\neg$, $\exists$ x, and finite disjunction
(3) if $\varphi(x)$ ε $L\langle A\rangle$ and t is a term then $\varphi(t)$ ε $L\langle A\rangle$
(4) If φ ε $L\langle A\rangle$ then every subformula of φ ε $L\langle A\rangle$

Subformulas are defined by recursion from the infinite disjunction by sub $(V\Phi) = \cup$ sub $(\varphi) \cup \{V\Phi$ taken over the set of formulas in Φ; with the basis defined for atomic formulas$\}$ by sub$(\varphi) = \{\varphi\}$; and for compound formulas by taking the union of the subformulas quantified or logically connected with the subformula relation applied to the original formula as a singleton. L is a countable language for first order logic. Let's call the fragment $L_{\omega 1,K}$.

Infinitary Language Categories-IFLCs

We take the Op category formed by $L_{\omega 1,B}$. Present a functor to take the Syntax Logische der Sprache from this category to the category Set, creating a limit corresponding to a model for the language defined from the syntax.te category. []

Op

A functor F: $L_{\omega 1,B} \rightarrow$ Set is defined by a list of sets Mn and functions fn. We refer to F by the generic model functor since it defines a generic model from language strings.

The functor F is a list of sets Mn corresponding to a free structure on $L_{\omega 1,B}$. The functions fn : Fn+1 $\rightarrow$ Fn. A generic model is defined by the limit on F.

IFLCs are not the same as a categorical interpretation for logic in the categorical logic sense. Thus IFLCs have their own theoretical properties and application areas defined by the author since July 1994 ECCT,European Category Theory Conference, Tours, France.

A model theory is defined to functors that are obtained from the exact ‘Logische Syntax der Sprache’ on the Infinitary language. Starting with the well-behaved countable fragment of an infinitary language $L_{\omega 1,\omega}$, we called $L_{\omega 1,K}$.

Generic Functors

Op

Define a functor F: $\mathbf{L\omega1,B}$ $\rightarrow$ **Set** by a list of sets Mn and functions fn.

Let us refer to the above functor by the **generic model functor** since it defines generic sets from language strings to form limits and models. The model theoretic properties are not defined in the present paper and are to be presented elsewhere by this author. The proof for the following theorem follows from the sort of techniques applied by the present author to define generic sets with the fragment $L_{\omega1,B}$ by this author 1982, 1983.

Theorem The generic model functor has a limit.

Theorem The Generic model functor defines a positive generic model.

Generic Functors and Models

Creating limits amounts to defining generic sets on $L_{\omega 1, \omega}$ for $L_{\omega 1,K}$. The techniques have been applied by this author to define Positive Forcing [1980's]. Let us refer to the above functor by the generic model functor since it defines generic sets from language strings to form limits and thus models.

The following theorem presented itself in the course f writing the proofs for creating functorial models on $-_{\omega 1,K}$. It is a consequence of the postive forcing author 1980-1981).

Theorem The power set of an inductive theory defined on $L_{\omega 1,K}$ generates a positive generic model for the theory, provided the inductive theory has a defined generic diagram for the canonical model.

String Models and Limits

The functor F is a list of sets F_n, consisting of

(a) the sets corresponding to an initial structure on

(b) $L_{\omega 1,B}$, for example,

to $f(t_1,t_2,\ldots t_n)$ in $L_{\omega 1,B}$ there corresponds the equality relation $f(t_1,\ldots,t_n)=ft_1\ldots t_n$ in Set;

(b) the functions $f_i:F_{i+1} \rightarrow F_i$. Form the product set $\Pi_i F_i$. Let $l = \{l_0,l_1,l_2,\ldots\}$, with each $l_n \in F_n$, and its projections $p_n: \Pi_i F_i \rightarrow F_n$, forming the following diagram.

$$\begin{array}{llllll}
F_0 \leftarrow & F_1 \leftarrow & & & F_n \leftarrow & F_{n+1} \leftarrow \\
\uparrow & & & & & \uparrow \\
\Pi_i F_i & \leftarrow & \leftarrow & \text{include} & \leftarrow & M = \text{Lim } F
\end{array}$$

To form a limit at a vertex for commuting with the projections take the subset S of the strings in S, which match under f, i.e. $f\, s_{n+1} = s_n$, for all n.

Robinson Consistency

- **Theorem** (Robinson's Consistency Theorem) Let $L1, L2$ be two languages. Let $L = L1 \cap L2$. Suppose T is a complete theory in L and $T1 \supset T$, $T2 \supset T$ are consistent in $L1, L2$, respectively. Then $T1 \cup T2$ is consistent in the language $L1 \cup L2$.

- We want to define functorial models piecemeal from language fragments by an infinite limit. There are two ways to view it .

- A- Take $L_{\omega 1, B}$ language fragments, define ω-chain models, and back and forth to a limit diagram model.
- B- Define models for the Fi from elementary diagrams, i.e
- define a limit model by embedding from a D<A,G> model.

- What complete theory can we fall onto? It has to be the theory
- Th(F), where F is the generic model functor, i.e.
- Op
- Th (F: $\mathbf{L}_{\omega 1, B} \rightarrow$ **Set**).

Elementary Chains

An elementary chain is a chain of models $A0 \subset A1 \subset A3 \subset A4 \subset \ldots A\beta$ such that $A\gamma < A\beta$ whenever $\gamma < \beta < \alpha$, $<$ is the elementary extension relation.

Elementary chains are applied to define Robinson’s consistency theorem and we plan to define similar techniques for defining String Models on the infinitary $L\omega1,B$ fragments. Of course the task is quite difficult and intricate functorial models on language fragments.

Fragment Consistency Models

OP

A functor **F**: **Lω1,B** → **Set** can be defined by sets **F**i, where the Fi's are defining a free structure on some subfragment of Lω1,B. To be specific we can define the subfragment models A(Fi) straight from the ω-inductive definition of the Infinitary fragment. F0 assigns names to the Set members, for example. F1 can define 1-place functions and relations, so on and so forth.

Taking the languages defined before with the Robinson's consistency theorem define Functorial Limit Chain models as follows. We shall refer to it by FLC-models.

FLK Models

- Let A and B, be models for Fi and Fi+1,respectively. Let A $\equiv_L$ B, and f: A $<_L$ B mean the L-reduct of A and B are elementarily equivalent and that f is an elementary embedding of A|L into B|L.

- Let A *FLC* model be the limit model defined by the elementary chain on the L-reducts of the models defined by the Fi's. A specific *FLK* model is defined by theorem 's proof.
- **Theorem** There is an elementary chain *FLK* model for $L_{\omega 1,K}$.

Fragment Model Towers

Let $T<i,i+1>$ be the complete theory in $L<i,i+1> = Li \cap Li+1$, defined by $Th(A(Fi) \cap A(Fi+1))$. Let Ti and $Ti+1$ be arbitrary consistent theories for the subfragments Li and $Li+1$, respectively, satisfying $Ti \supset T<i,i+1>$ and $Ti+1 \supset T<i,i+1>$. Let Ai be $A(Fi)$ and Bi be $A(Fi+1)$.

Starting with a basis model $A0$, $A0|L<0,1>$ and $B0|L<0,1>$ are models of a complete theory, hence, $A0|L<0,1> \equiv B0|l<0,1>$. It follows that the elementary diagram of $A0|l<0,1>$ is consistent with the elementary diagram of $B0|L<0,1>$. Therefore, there are elementary extensions $B1 > B0$ and an embedding $f1: A0 < B1$ at $L<01>$. Passing to the expanded language $L<0,1>A0$, we have $(A0,a)a \varepsilon A0 \equiv L<0,1> A0\ (B1,fa)\ a\ \varepsilon\ A0$. g1 inverse is an extension of f1.

Iterating, we obtain the tower depicted sideways

$A0 < A1 < A2 < \ldots\ldots\ldots$

$\backslash f1\ |g1\ \backslash f2\ |g2$

$B0 < B1 < b2 < \ldots\ldots$

The Alpine Language Slalom

Slalom between the language pairs $L_i\ L_{i+1}$ gates to a limit model for L. For each m, fm $\subset$ inverse(gm) $\subset$ fm+1, fm: $A_{m+1} < B_m$ at $L_{<m+1,m>}$. Let $A = \cup A_m,\ m < \omega$, $B = \cup B_m,\ m < \omega$. a B is isomorphic to a model B' such that $A|L = B'|L$. Piecing A and B' together we obtain a model M for $L_{\omega 1,B}$.

Note that the language L varies in our proof, while it does not in the ω-chain model proof for Robinson's consistency theorem. Thus the techniques are perhaps new.

A concluding theorem.

Theorem There are functorial elementary chain models.

- Slalom between the language pairs L_i, L_{i+1} gates to a limit model for L. For each m, $f_m \subset \text{inverse}(g_m) \subset f_{m+1}$, $f_m: A_{m-1} < B_m$ at
- $L\langle m-1,m\rangle$.

- Let $A = \cup A_m$, $m < \omega$, $B = \cup B_m$, $m < \omega$. B is isomorphic to a model B' such that A|L = B'|L. Piecing A and B' together we obtain a model M for $L_{\omega 1,B}$. []

- Note that the language L varies in our proof, while it does not in the ω-chain model proof for Robinson's consistency theorem. Thus the techniques are perhaps new.

Fragment Consistency

The following is our infinitary complement to Robinson Consistency

Theorem Frgement Model Consistency

Let T<i,i+1> be the complete theory in L<i,i+1> = Li intersect Li+1, defined by Th (A(Fi) intersect A(Fi+1)). Let Ti and Ti+1 be arbitrary consistent theories for the subfragments Li and Li+1, respectively, satisfying Ti contains T<i,i+1> and Ti+1 contains T<i,i+1>. Let Ai be A(Fi) and Bi be A(Fi+1). Starting with a basis model A0, A0 | L<0,1> and B0 | L<0,1> are models of a complete theory, hence, A0 | L<0,1> $\equiv$ B0 | <0,1>. It follows that the elementary diagram of A0|l<0,1> is consistent with the elementary diagram of B0 | L<0,1>.

(i) There are iterated elementary extensions Bi+1> Bi and an embedding fi: Ai < Bi

(ii) A slalom between the language pairs Li Li+1 gates to a limit model for L obtains a model M for $L\omega_{1,B}$.

Fragment Horn Models

- Positive forcing had defined T* to be T augmented with induction schemas on the generic diagram functions.
- **Proposition** Let I be the set T*. Let φ (x1…xn) be a Horn formula and let $\mathfrak{R}$i i ε I be models for language L. let a1….an ε Πi εI Ai. The $\mathfrak{R}$i are fragment Horn models.
- If {i ε I: $\mathfrak{R}$i ⊨ φ[(a1(i) …an(i)]} then the direct Π_D on $\mathfrak{R}$i ⊨ φ[a1D …anD], where D is the generic filter on T*.

Fragment Consistent Models over Algebras

- Applying the above definitions for positive formulas we can prove the following as a lemma and a consequent theorem.
- **Theorem** The embedding to form elementary chains on Fi's can be defined by a back and forth model design from the strings in $L_{\omega 1,K}$ language fragments.
-
- **Theorem** By defining models corresponding to the Fi on the fragments as an ω-chain from the elementary diagrams on the Th (A(Fi)) a generic model is defined by the limit.

Filters

- Let I be a nonempty set. Let S(I) be the set of all subsets of I. A filter D over I is defined to be a set D < S(I) such that
  I ε D; if X,Y ε D, hen X ∩ Y ε D,; if X ε D and X < Z < I, then Z ε D.
  Note that every filter D is a nonempty set since I ε D. exmpale filters are the trivial filter D = {I}. The improper filter D = S(I).
  For each Y < I, eh filter D = {X <I; Y < X}; this filter is called the principal filter generated by Y. D is said to be aproper filter iff it is not the improper filter S(I).

- Let E be a subset of S(I). By the filter generated by E we mean the intersection D of of all filters over I which include E: D = ∩ {F: E < F and F is a filer over I}. E is said to have the finite intersection property iff the intersection of any finite nuber if elements of E is nonempty.

- Can prove that the filter D generated by E, any subset E of S(I), is a filter over I.

Forcing Properties

- **Definition** A forcing property for a language L is a triple $\mathscr{F} = (S, \leq, f)$ such that
- (i) $(S, \leq)$ is a partially ordered structure with a least element, for example, 0;
- (ii) f is a function which associates with each p in S a set f(p) of atomic sentences of L[C];
- (iii) whenever $p \leq q$, $f(p) \subseteq f(q)$;
- (iv) let l and t be terms of L[C] without free variables and p in S; let φ be a formula of L[C] with one free variable. Then if $(l \neg = t)$ is in f(p) then (t=l) is in f(q) for some $q \geq p$. $\varphi(t)$ in f(p) implies $\varphi(l)$ is in f(q) for some $q \geq p$. For some c in C and $q \geq p$, (c=l) is in f(q). The elements of S are called conditions for F.

- **Definition** The relation $p \Vdash + \varphi$ read "p positively forces φ" is defined for conditions p an q in S as follows: for an atomic sentence φ, $p \Vdash + \varphi$ iff φ is in p; for an open formula φ of the form f(X) = g(X), $p \Vdash + \varphi$, iff for all c in L[C], where c is an n-tuple of constants form L[C] and all q, $p < q$ implies there is an r such that $r > q$ and $r \Vdash + f(c) = g(c)$. For arbitrary sentence φ, $p \Vdash + \varphi$ iff p +forces a universal fromula logically equivalent to φ.

- **Definition** Let $\mathscr{F} = (S,\leq,f)$ be a positive forcing property, we say that a subset G of S is positive generic iff
-
- (i) p in G and $q < p$ implies q in G;
- (ii) p,q in G, implies there is an r in G such that $p \leq r$ and $q \leq r$;
- (iii) for each sentence φ, there exists p in G, such that either $p \Vdash^{+} \varphi$ or there is no q in S, $q > p$ such that $q \Vdash^{+} \varphi$.
- A special case of the above definition is when S consists of sets of formulas. For such S , we can make a substitution: subset relation for <. Clause (iii) can then be stated:
- (iii)' for each sentence φ, there exists p in G such that either $p \Vdash^{+} \varphi$ or $p \cup \{\varphi\}$ is not a condition for $\mathscr{F}$.

Infinitary Consistency Over Algebras

- **Theorem** Infinitary Fragment Consistency on Algebras Let T<i,i+1> be complete theories for L<i,i+1> = Li intersect Li+1. Let Ti and Ti+1 be arbitrary consistent positive theories for the subfragments Li and Li+1, respectively, satisfying Ti contains T<i,i+1> and Ti+1 contains T<i,i+1>. Let Ai be A(Fi) and Bi be A(Fi+1).

- (i) There are iterated elementary extensions Bi+1> Bi and an embedding fi: Ai < Bi

- (ii) Slalom between the language pairs Li Li+1 gates to a limit model for L, a model M for $L_{P,\omega}$.

Positive Morphisms and Models

- **Definition** A formula is said to be positive iff it is built from atomic formulas using only the connectives &, v and the quantifiers $\forall$, $\exists$.

- **Definition** A formula φ ($x_1, x_2, \ldots, x_n$) is preserved under homomorphisms iff for any homomorphisms f of a model A onto a model B and all $a_1, \ldots, a_n$ in A if A $\models$ [$a_1, \ldots, a_n$] B $\models$ [$fa_1, \ldots, fa_n$].

- **Definition** A generic diagram for a structure M is a diagram D<A,G>, such that there is a proper diagram defined with a specific function set.
- The function set might be Σ_1 Skolem functions for the set theory example.
- Positive forcing had defined T* to be T augmented with induction schemas on the generic diagram functions.

- **Proposition** Let I be the set T*. Let φ ($x_1 \ldots x_n$) be a Horn formula and let $\Re_i$ $i \in I$ be models for language L. let $a_1 \ldots . a_n \in \Pi_{i \in I} A_i$. The $\Re_i$ are fragment Horn models.
- If $\{i \in I: \Re_i \models \varphi[(a_1(i) \ldots a_n(i)]\}$ then the direct ΠD on $\Re_i \models \varphi[a_1 D \ldots a_n D]$, where D is the generic filter on T*.

Consistency on the Positive

- Let us start from certain model-theoretic premises with the follwing two propositions known form basic model theory.

  Proposition Let $\mathfrak{R}$ and |B be models for L. Then $\mathfrak{R}$ is isomorphically embedded in |B iff |B can be expanded to a model of the diagram of $\mathfrak{R}$.

- **Proposition** Let $\mathfrak{R}$ and |B be models for L. Then $\mathfrak{R}$ is homomrphically embedded in |B iff |B can be expanded to a model of the positive diagram of $\mathfrak{R}$. Let Σ be a set of formulas in the variables $x1 \ldots xn$. Let $\mathfrak{R}$ be a model for L. We say that $\mathfrak{R}$ realizes Σ iff some n-tuple of elements of A satisfies Σ in $\mathfrak{R}$.$\mathfrak{R}$ omits Σ iff $\mathfrak{R}$ does not realize Σ. Form basic model theory we know that

  Let Σ $(x1 \ldots xn)$ be a set of formulas of L. A theory T in L is said to locally realize Σ iff there is a formula φ $(x1 \ldots xn)$ in L s.t. φ is consitent with T(ii) for all $\sigma \in \Sigma$, $T \models \varphi \rightarrow \sigma$
  That is every n-tuple of T which realizes φ satisfies Σ.We say that T locally omits Σ iff T does not locally realize Σ.For our purposes we define a new realizability basis.

Positive Consistency

- **Definition** Let Σ (x1…xn) be a set of formulas of L. Say that a positive theory T in L *positively locally realize* Σ iff there is a formula φ (x1…xn) in L s.t. (i) φ is consitent with T(ii) for all $\sigma \varepsilon \Sigma$, $T \models \varphi$ or $T \cup \sigma$ is not consistent.

- **Defintion** Given models A and B, with generic diagrams D_A and D_B we say that DA homomrphially extends D_B iff there is a homomorhic emdedding f: A → B.

- **Theorem** Let *L1, L2* be two positive languages. Let *L* = *L1* ∩ *L2.* Suppose T is a complete theory in *L* and T1 ⊃ T, T2 ⊃ T are consistent in *L1, L2,* respectively. Suppose there is model M definable from a positive diagram in the language *L1* ∪ *L2 such that* there are models M1 and M2 for T1 and T2 where M can be homorphically embedded in M1 and M2.
- (i) T1 ∪ *T2 is consistent.*
- (ii) There is model N for T1 ∪ *T2* definable from a positive diagram that homomorphically extends that of M1 and M2.

Positive Fragment Categories

- Let $L_{P,\omega}$ be the positive fragment obtained from the Kiesler fragment.
- Define the category $L_{P,\omega}$ to be the category with objects positive fragments and arrows the subformula preorder on formulas.
- Define a functor $F: L_{P,\omega}^{Op} \rightarrow$ Set by a list of sets Mn and functions fn. The functor F is a list of sets Fn, consisting of
- (a) the sets corresponding to an initial structure on LP,ω, for example the free syntax tree structure, where to f(t1,t2,.. tn)
- in $L_{P,\omega}$ there corresponds the equality relation f(t1,...,tn)=ft1...tn in Set;
- (b) the functions fi:Fi+1 $\rightarrow$ Fi.

Generic Filter Models

- **Definition 4.10** Let Q be a poset, F a family of sets, G subset of Q. the G is F-generic iff the following holds;
- (i) whenever p in G and $q < p$, then q is in G.
- (ii) whenever p,q in G, then there exists an r in G with $r > p$ and $r > q$.
- In particular, any two elements of G are compatible.
- (iii) $G \cap D$ -= $\emptyset$ for any dense D subset of Q with D in F.
- Note that a subset D of Q is said to be dense iff for every p in Q there is
- a q in D with $p < q$.

- **Theorem 4.5** $\mathcal{O}$(T*) is generating a generic model with the F-generic filter.

Positive Forcing and Filters

- **Theorem** (Nourani 1981) The positive forcing T^* is a F-generic filter.

- F-genericity is essential to the Martin's axiom statement.

Generic Products

- **Definition** A formula is said to be positive iff it is built from atomic formulas using only the connectives &, v and the quantifiers $\forall$, $\exists$.

- **Definition** A formula φ (x1,x2,...,xn) is preserved under homomorphisms iff for any homomorphisms f of a model A onto a model B and all a1,...,an in A if A |= [a1,...,an] B |= [fa1,...,fan].

- **Theorem** A consistent theory is preserved under homorphisms iff T has a set of positive axioms.

Horn Product Models

- **Known from basic models theory are the following propositions:**
- **Proposition** Let φ be a universal sentenece. Then φ is a (finite) direct product sentence iff φ is equivalent to a universal Horn senetence.
- **Proposition** Let φ be an existential sentenece. Then φ is a (finite) direct product sentence iff φ is equivalent to an xistential Horn senetence.
- **Propostion** Let φ be a universal sentence. Then φ is a (finite) direct product sentence if and only if φ is equivalent to a universal Horn sentence.
- **Proposition** Let φ be an existential sentence. Then φ is a (finite) direct product sentence if φ is equivalent to an existential Horn sentence.

Horn Completion

- **From the author's *Functional generic filter,* ASL, Montreal, May 2006.**

- **Theorem** Assume the continum hypothesis $2^{\omega} = \omega +$, then φ is completable in T^* iff φ is equivalent to a universal Horn sentence.
-
- **However, the completability theorem at T^* on universal Horn sentences might be provable without CH: $2\omega = \omega +$, therefore the above might apply without CH.**

- **From the propostion on filters and direct products, let $\Re i$ $i \varepsilon I$ be models for language L.**

Filters and Direct Products

- **Lemma** Positive forcing T* is a principal proper filter.

- **Theorem** Let I be the set T*. Let φ (x1…xn) be a Horn formula and let $\Re$i i ε I be models for language L. let a1….an ε ΠiεI Ai.
  If {i ε I: $\Re$i |= φ[(a1(i) …an(i)]} then the direct product over D on $\Re$i $\models$ φ[a1D …anD], where D is the generic filter on T*.

- Remark: The $\Re$i are fragment Horn models.

Horn Filters

- **Theorem** Let $(P, \leq)$ be a positive Horn poset and $p \in P$. If D is a countable family of dense subsets of P then $T^*(P)$ is D-generic filter F in P *such that* $p \in F$ and every $p \in F$ has a positive local realization.
- The above theorem is a Horn density counterpart to the Rasiowa-Sikorski lemma.

Rasiowa-Sikorski

- Rasiowa-Sikorski lemma is very important to sets and foundations. The author was pointed to the lemma by an elder sets colleague a few months ago, not having ever examined that, when he had contemplated a functorial program to set theory.

- The above theorem was instantaneous when the lemma was examined.

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